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Interaction of two solitary waves in a ferromagnet

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Abstract. A type of solitary wave in a ferromagnet is found by a multiscale expansion method; it obeys the completely integrable Korteweg–de Vries equation. The interaction between a wave of this propagation mode and another known mode that also allows soliton propagation is studied. The equations describing the interaction are derived using a multiscale expansion method and then reduced to an integral form, and solved explicitly for particular initial data for which one of the waves can be considered as a soliton. A phase shift of this soliton appears. Transmission and reflexion coefficients are computed for the second wave.

1. *Introduction and generalities*

Propagation of electromagnetic waves in ferromagnets has generated much interest, especially in relation to ferrite devices at microwaves frequencies [1]. This subject was extensively studied from the linear viewpoint in the 1950s and 1960s [2] and in connection with the theory of ferromagnetic resonance [3, 4]. However, this problem is intrinsically nonlinear. Development of soliton theory and the multiscale expansion method [5, 6] allowed new studies to be completed [7]. Recently, using these methods, Nakata [8] discovered a type of solitary wave that is governed by the modified Korteweg–de Vries (mKdV) equation.

Following this work I studied nonlinear wave propagation in ferrites, especially the focusing or defocusing of a quasi-monochromatic plane wave that is described by the nonlinear Schrödinger (NLS) equation [9, 10]. I also studied the resonant interaction of three waves [11]. As I was seeking the equations describing the non-resonant interaction of two quasi-monochromatic waves, an interesting feature arose: the 'constant' term in the Fourier series expansion of the waves appeared to be very important, allowing solitary waves to propagate and interact with fast oscillating waves.

This paper intends to clarify the properties of these solitary waves and is organized as follows. In section 2 we recall Nakata's results, using our notation and adding some comments and personal observations that will be useful for comprehending the rest of the paper. In section 3 we describe a propagation mode that is governed by the Korteweg–de Vries (KdV) equation. In section 4 we derive the equations that govern the interaction between the two solitary waves. Section 5 is devoted to a resolution of this equation. A qualitative conclusion is given in section 6.

2. The mKdV *mode*

A mode of nonlinear solitary waves in ferromagnets was discovered by Nakata in 1990 [8]. He used a classical model in which the evolution of the magnetic field $\boldsymbol{H}$ and the magnetization density M is described by the following equations:

$$-\nabla(\nabla\cdot \boldsymbol{H})+\Delta \boldsymbol{H}=\frac{1}{c^2}\partial_t^2(\boldsymbol{H}+\boldsymbol{M}) \tag{1}$$

$$\partial_t \boldsymbol{M}=-\delta\mu_0 \boldsymbol{M}\wedge \boldsymbol{H}. \tag{2}$$

Throughout this paper, we will use the notation $\partial_x u$ for the partial derivative of u with respect to the variable x. δ is the gyromagnetic ratio, and $c=1/\sqrt{\mu_0\tilde{\varepsilon}}$ is the speed of light based on the dielectric constant of the ferromagnet. Indeed, it is assumed that the electric field $\boldsymbol{E}$ and the electric induction $\boldsymbol{D}$ are related by the linear relation $\boldsymbol{D}=\tilde{\varepsilon}\boldsymbol{E}$.

This model is commonly used to describe wave propagation in ferrites, particularly at microwave frequencies [1, 2]. The model neglects the finite-size effects of the inhomogeneous exchange interaction, of the sample, and the efforts of damping. For the sake of simplicity, we rescale $\boldsymbol{M}, \boldsymbol{H}, t$ into $\mu_0\delta/c\boldsymbol{M}$, $\mu_0\delta/c\boldsymbol{H}$, ct, respectively, and obtain the equations

$$-\nabla(\nabla\cdot \boldsymbol{H})+\Delta \boldsymbol{H}=\partial_t^2(\boldsymbol{H}+\boldsymbol{M}) \tag{3}$$

$$\partial_t \boldsymbol{M}=-\boldsymbol{M}\wedge \boldsymbol{H}. \tag{4}$$

As in [8], let us first derive the dispersion relation corresponding to the system (3) and (4). The system is linearized about a constant solution

$$\boldsymbol{M}_0=\boldsymbol{m}=\begin{pmatrix} m_x\\ m_t\\ 0\end{pmatrix}=\begin{pmatrix} m\cos\varphi\\ m\sin\varphi\\ 0\end{pmatrix} \tag{5}$$

$$\boldsymbol{H}_0=\alpha\boldsymbol{m} \tag{6}$$

and we look for solutions proportional to $\exp \mathrm{i}(kx-\omega t)$. The dispersion relation reads as follows.

$$\left(1+\alpha\left(1-\frac{k^2}{\omega^2}\right)\right)^2 m^2\cos^2\varphi+(1+\alpha)\left(1-\frac{k^2}{\omega^2}\right)\left(1+\alpha\left(1-\frac{k^2}{\omega^2}\right)\right)m^2\sin^2\varphi$$
$$=\left(1-\frac{k^2}{\omega^2}\right)^2\omega^2. \tag{7}$$

This relation presents three branches of which two approach the point $k=0$, $\omega=0$ with a finite slope $u=\omega/k$. The two values of this phase velocity u are

$$u_1=\sqrt{\frac{\alpha}{1+\alpha}} \tag{8}$$

and

$$u_2=\sqrt{\frac{\alpha+\sin^2\varphi}{1+\alpha}}. \tag{9}$$

For very small values of ω and k with a finite value of $u=\omega/k$, the oscillating wave (described by the solution $\mathrm{e}^{\mathrm{i}(kx-\omega t)}$) gives rise to a solitary wave solution (of the form $f(kx-\omega t)$ with f vanishing at infinity). The nonlinear behaviour of the solitary wave corresponding to u_1 has been studied by Nakata, the wave with velocity u_2 is studied in section 3 of this paper.

The expansion of ω/k about the value u_1 in a power series of k, assumed to be small, suggests that we introduce the stretched variables

$$\begin{cases}\xi=\varepsilon(x-Vt)\\ \tau=\varepsilon^3 t\end{cases} \tag{10}$$

where ε is a small parameter. $\boldsymbol{M}$ and $\boldsymbol{H}$ are expanded in a power series of ε:

$$\begin{cases} \boldsymbol{M} = \boldsymbol{M}_0 + \varepsilon \boldsymbol{M}_1 + \varepsilon^2 \boldsymbol{M}_2 + \cdots \\ \boldsymbol{H} = \boldsymbol{H}_0 + \varepsilon \boldsymbol{H}_1 + \varepsilon^2 \boldsymbol{H}_2 + \cdots \end{cases} \tag{11}$$

and the quantities $\boldsymbol{M}_j$ and $\boldsymbol{H}_j$ are assumed to be functions of the stretched variables ξ and τ.

The boundary conditions are

$$\begin{cases} \boldsymbol{M}_0 \underset{\xi\to-\infty}{\longrightarrow} \boldsymbol{m} \\ \boldsymbol{H}_0 \underset{\xi\to-\infty}{\longrightarrow} \alpha \boldsymbol{m} \end{cases} \tag{12}$$

and

$$\begin{cases} \boldsymbol{M}_j \underset{\xi\to-\infty}{\longrightarrow} 0 \\ \boldsymbol{H}_j \underset{\xi\to-\infty}{\longrightarrow} 0 \end{cases} \tag{13}$$

for $j \neq 0$. They assume that, at a large distance from the region where the wave is localized, the fields tend to the static state (5) and (6). Practically, the sample of ferrite is immersed in a constant field $\boldsymbol{H}_{\text{ext}}$. The corresponding magnetic field $\alpha \boldsymbol{m}$ inside the sample depends on the shape of this constant field through demagnetizing factors. As we are modelling the sample by an infinite medium, we ignore the relationship between $\boldsymbol{H}_{\text{ext}}$ and $\alpha \boldsymbol{m}$. Thus we consider the magnetic field as given and we call it an exterior field unless it differs from $\boldsymbol{H}_{\text{ext}}$. Notice also the very important fact that $\boldsymbol{H}_0$ and $\boldsymbol{M}_0$ are not assumed to be constant.

Let us expose the calculus in a slightly different way to [8]. At order ε^0, it is found that

$$\boldsymbol{M}_0 \wedge \boldsymbol{H}_0 = 0 \tag{14}$$

$$\partial_\xi^2 (H_0^x + M_0^x) = 0 \tag{15}$$

and for $s = y, z$

$$\partial_\xi^2 (\gamma H_0^s + M_0^s) = 0. \tag{16}$$

The constant γ is defined by

$$\gamma = 1 - \frac{1}{V^2}. \tag{17}$$

Equation (14) can now be written as

$$\boldsymbol{H}_0 = \lambda(\xi, \tau) \boldsymbol{M}_0 \tag{18}$$

where λ is an arbitrary scalar function. We integrate equations (15) and (16) using the boundary conditions (12), and find, using (18), that

$$\begin{cases} (1+\lambda) M_0^x = (1+\alpha) m_x \\ (1+\lambda\gamma) M_0^y = (1+\alpha\gamma) m_t \\ (1+\lambda\gamma) M_0^z = 0. \end{cases} \tag{19}$$

In this system, either $1 + \alpha\gamma = 0$, or not. In the former case, we must have $1 + \lambda\gamma \equiv 0$, otherwise $M_0^y = M_0^z = 0$, and we can verify that there is only the trivial solution. Thus, $\lambda \equiv \alpha$ and $V = \sqrt{\alpha/1+\alpha}$. This is the case studied by Nakata. Another solution is found when we do not assume $1 + \lambda\gamma \equiv 0$. This case will be studied in the next section.

For the above value of V, system (18) and (19) reduces to

$$H_0 = \alpha M_0 \tag{20}$$

$$M_0^x = m_x. \tag{21}$$

At order ε^1, it is found that there exists a function $\theta = \theta(\xi, \tau)$ such that

$$\begin{cases} M_0^y = m_t \cos\theta \\ M_0^z = m_t \sin\theta \end{cases} \tag{22}$$

with $\lim_{\xi \longrightarrow -\infty} \theta = 0$. At order ε^2, we obtain the expression

$$M_1^x = -H_1^x = \frac{V}{1+\alpha} \partial_\xi \theta \tag{23}$$

and the equation verified by θ which reads

$$\partial_\tau \theta + \tfrac{1}{2} J (\partial_\xi \theta)^3 + J \partial_\xi^3 \theta = 0 \tag{24}$$

where the constant J (called μ in [8]) is given by

$$J = \frac{\alpha^{5/2}}{2(1+\alpha)^{7/2} \sin^2 \varphi}. \tag{25}$$

If we put

$$f = \partial_\xi \theta \tag{26}$$

we find

$$\partial_\tau f + \tfrac{3}{2} J f^2 \partial_\xi f + J \partial_\xi^3 f = 0 \tag{27}$$

which is the mKdV equation. It is known that this equation is completely integrable by the inverse scattering transform (IST) method, and related to the KdV equation by the Miura transform [12, 13].

The one-soliton solution of (27) reads

$$f = \frac{2a}{\cosh a(\xi - p\tau)} \tag{28}$$

with $a^2 = p/J$ and p an arbitrary positive constant. θ is then given by

$$\cos\theta = 1 - \frac{2}{\cosh^2 a(\xi - p\tau)} \tag{29}$$

where θ increases from 0 to 2π or decreases from 2π to 0, depending on the sign of a. Notice that the function f, which obeys the mKdV equation (27), can be considered as the derivative of the angle θ (the angle of precession of the whole magnetization density vector around the propagation direction) and can also be considered as the amplitude of the first-order term M_1^x of the longitudinal component of the magnetization density. Equation (27) describes a weakly nonlinear phenomenon, thus the amplitude of the wave should be small. A peculiar feature is that the varying function θ is of order zero, and that all the magnetization density, with its saturation norm, rotates. However, such a rotation does not require a large amount of energy if the angle between two consecutive spins is small enough, that is, if the typical variation length of the wave is large enough. This is the hypothesis corresponding to the use of the slow variables (10). The fact that the function f, which is governed by the mKdV equation, is precisely the derivative of θ, which compares to ε, also corresponds to this condition.

This propagation mode has been introduced as a limiting case of small-amplitude waves, but it is, in fact, of finite amplitude. It is known that finite-amplitude waves can propagate in a ferromagnet [7, section 2.1]. Using our notation, the corresponding solution of system (3) and (4) is given by

$$M = m \begin{pmatrix} \cos\varphi \\ \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \end{pmatrix} \tag{30}$$

$$H = \begin{pmatrix} \alpha m\cos\varphi - \omega \\ \alpha m\sin\varphi\cos\theta \\ \alpha m\sin\varphi\sin\theta \end{pmatrix} \tag{31}$$

where m, α, and φ are constants, and

$$\theta = kx - \omega t \tag{32}$$

$$\frac{\omega}{k} = \pm\sqrt{\frac{\alpha}{1+\alpha}}. \tag{33}$$

Comparing expressions (30) and (31) with (20)–(22), we see that Nakata's solitonic mode can be interpreted as a limiting case of this solution for small values of ω and k.

3. A propagation mode governed by the KdV equation

The dispersion relation (7) has two branches with a finite slope $u = \omega k$ at the origin. For one of these branches we have just seen that in the long-wave approximation, the waves are governed by the mKdV equation. The question that naturally arises is: what do the waves that belong to the second branch look like. We intend to give an answer in this section.

Where we gave equations (18) and (19) of the perturbation scheme at order ε^0, we left aside some solutions. Under the same hypothesis, we can prove that the function λ must be constant. If we take the scalar product of the basic equation (4) with M, we find that

$$M \cdot \partial_t M = 0. \tag{34}$$

Thus, $\|M\|$ is constant. At order ε^0, this implies that $\|M_0\|$ is constant. If $1 + \alpha\gamma \neq 0$ then $1 + \lambda\gamma$ is always non-zero, and we can write the condition $\|M_0\|^2 \equiv m_x^2 + m_t^2$ as

$$(1+\alpha)^2 m_x^2 (1+\lambda\gamma)^2 + (1+\alpha\gamma)^2 m_t^2 (1+\lambda)^2 = (m_x^2 + m_t^2)(1+\lambda\gamma)^2(1+\lambda)^2 \tag{35}$$

λ is a solution of this fourth-order polynomial equation and, thus, is constant and equal to α. It follows immediately that in this case H_0 and M_0 are constant.

Still using the stretching expression (10) and the expansion (11), we find at order ε^1 that

$$M_1 = m_1 g \qquad m_1 = \begin{pmatrix} -\mu m_x \\ -\gamma(1+\alpha)m_t \\ 0 \end{pmatrix} \tag{36}$$

$$H_1 = h_1 g \qquad h_1 = \begin{pmatrix} \mu m_x \\ (1+\alpha)m_t \\ 0 \end{pmatrix} \tag{37}$$

where

$$\mu = 1 + \alpha\gamma \tag{38}$$

and $g = g(\xi, \tau)$ is an unknown function.

At order ε^2, the condition needed for the function g to differ from 0 is

$$\mu m_x^2 + \gamma(1+\alpha)m_t^2 = 0 \tag{39}$$

that is

$$V = \sqrt{\frac{\alpha + \sin^2\varphi}{1+\alpha}}. \tag{40}$$

We compute $\boldsymbol{H}_2$ and $\boldsymbol{M}_2$ in terms of a second unknown function f, and h. The compatibility condition for equation (4) at order ε^3 leads to an equation for g of the form

$$\partial_\xi(g^2) = pg^2 \tag{41}$$

where p is a real constant. (41) has no non-trivial bounded solution, thus $g = 0$. Thus, $\boldsymbol{H}_1 = 0$ and $\boldsymbol{M}_1 = 0$.

This requirement could also have been found in the following way. We intend to find a propagation mode described by the KdV equation which reads

$$\partial_\tau g + Ag\partial_\xi g + B\partial_\xi^3 g = 0. \tag{42}$$

The scaling that we use must be consistent with the homogeneity properties of this equation, thus the transformation

$$\begin{cases} \hat{g} = \varepsilon^l g \\ \xi = \varepsilon^m X = \varepsilon^m(x - Vt) \\ \tau = \varepsilon^n t \end{cases} \tag{43}$$

must give a result independent of the ε used in equation (42). We find

$$l + n = 2l + m = l + 3m. \tag{44}$$

Thus, choosing $m = 1$, $l = 2$ and $n = 3$, we obtain the scaling (10) and (11), with $\boldsymbol{H}_1 = \boldsymbol{M}_1 = 0$.

Thus, it is the term of order ε^2 in the expansion of the fields that will be described by the nonlinear equation (KdV). This can be interpreted in the following sense. Observed at the same time and space scales as previously, the propagation mode considered here will have the behaviour described by the KdV equation (formation of solitons, and so on) for much smaller wave intensities as does the previous mode. For higher intensities, the weakly nonlinear approximation will no longer be valid, and nonlinear effects of higher order are expected. Thus, we have

$$\boldsymbol{M}_2 = \boldsymbol{m}_1 g \qquad \boldsymbol{H}_2 = \boldsymbol{h}_1 g \tag{45}$$

with $\boldsymbol{h}_1$, $\boldsymbol{m}_1$ given by (36) and (37). At order ε^3, we obtain the compatibility condition (39), the velocity V given by (40), and the expression

$$\boldsymbol{H}_3 = \boldsymbol{h}_1 f - V\frac{m_x}{m_t} e_z \partial_\xi g \tag{46}$$

where f is an unknown function and

$$\boldsymbol{M}_3 = \boldsymbol{m}_1 f + \gamma V \frac{m_x}{m_t} e_z \partial_\xi g. \tag{47}$$

We call e_x, e_y, e_z the vectors of the reference frame.

The compatibility condition at order ε^4 is trivial and we find

$$M_4^x = -H_4^x = -\mu m_x \psi \tag{48}$$

$$H_4^y = (1+\alpha)m_t\psi + \frac{\gamma V^2}{\mu m_t}\partial_\xi^2 g + \frac{2\alpha(1+\alpha)m_t}{\mu V^3}\int_{-\infty}^{\xi}\partial_\tau g + \frac{(1+\alpha)m_t}{V^2}g^2 \tag{49}$$

$$M_4^y = -\gamma H_4^y + \frac{2(1+\alpha)m_t}{V^3}\int_{-\infty}^{\xi}\partial_\tau g \tag{50}$$

$$M_4^z = -\gamma H_4^z = \gamma V \frac{m_x}{m_t}\partial_\xi f. \tag{51}$$

At order ε^5, equation (4) reads

$$-V\partial_\xi \boldsymbol{M}_4 + \partial_\tau \boldsymbol{M}_2 = -(\boldsymbol{M}_0 \wedge \boldsymbol{H}_5 + \boldsymbol{M}_2 \wedge \boldsymbol{H}_3 + \boldsymbol{M}_3 \wedge \boldsymbol{H}_2 + \boldsymbol{M}_5 \wedge \boldsymbol{H}_0). \tag{52}$$

Taking the dot product of (52) with $\boldsymbol{m}$, we obtain the compatibility condition

$$-V\partial_\xi \boldsymbol{m} \cdot \boldsymbol{M}_4 + \partial_\tau \boldsymbol{m} \cdot \boldsymbol{M}_2 = \boldsymbol{m} \cdot (\boldsymbol{M}_2 \wedge \boldsymbol{H}_3 + \boldsymbol{M}_3 \wedge \boldsymbol{H}_2). \tag{53}$$

Using the results of the preceding orders, we reduce this equation to the KdV equation (42), with

$$A = \frac{3}{2}\frac{\mu V}{\alpha} = \frac{3}{2\alpha}\sqrt{\frac{1+\alpha}{\alpha+\sin^2\varphi}}\sin^2\varphi \tag{54}$$

$$B = \frac{\gamma V^5}{2\alpha(1+\alpha)m_t^2} \tag{55}$$

$$B = -\frac{1}{2m^2}\frac{(\alpha+\sin^2\varphi)^{3/2}}{\alpha(1+\alpha)^{7/2}}\frac{\cos^2\varphi}{\sin^2\varphi}. \tag{56}$$

The KdV equation (42) describes weakly nonlinear dispersive waves in various branches of physics. This has been extensively studied, is completely integrable by the IST method [14], and admits soliton solutions that are peculiarly stable. A description of the solution of the Cauchy problem for the KdV equation in terms of its solitonic components has been tested in the case of water waves and it gives really good accuracy in relation to the experimental data; [15] gives an abundant bibliography.

Let us write the soliton solution of equation (42):

$$g(\xi,\tau) = \frac{12k^2B}{A}\frac{1}{\cosh^2 k(\xi - 4k^2B\tau)} \tag{57}$$

where k is an arbitrary real constant. The soliton velocity is proportional to B and its amplitude to B/A. As the angle φ between the propagation direction and the exterior field tends to zero, A tends to zero and B tends to infinity. Thus, the soliton velocity and its amplitude become large for propagation nearly parallel to the exterior field. As φ tends to $\pi/2$, A tends to the finite limit $3/2\alpha$, and B tends to zero. Both the soliton velocity and amplitude become small in this case. The limit equation is

$$\partial_\tau g + \frac{3}{2\alpha} g \partial_\xi g = 0. \tag{58}$$

This equation is solvable by the method of characteristics, and leads to shock solutions [16]. The discontinuity of these shock solutions disappears when taking into account higher-order nonlinear terms, or damping. A Burgers' equation has been derived from the same model with the addition of a damping term and for the same propagation mode but at other time and space scales. Its coefficients disappear as $\varphi \to \pi/2$ [17].

Note that, in contrast with the mKdV mode, the present solitonic mode is of small amplitude. Let us linearize system (3) and (4), as at the beginning of section 2. The eigenvectors obtained for the field $\boldsymbol{H}$ and the magnetization $\boldsymbol{M}$ are

$$\boldsymbol{h}'_1 = \begin{pmatrix} \mathrm{i}\gamma'\mu' m_t \\ -\mathrm{i}\mu' m_x \\ \gamma'\omega \end{pmatrix} \tag{59}$$

$$\boldsymbol{m}'_1 = \begin{pmatrix} -\mathrm{i}\gamma'\mu' m_t \\ \mathrm{i}\gamma'\mu' m_x \\ -\gamma'^2\omega \end{pmatrix} \tag{60}$$

with

$$\gamma' = 1 - \frac{k^2}{\omega^2} \qquad \mu' = 1 + \alpha\gamma'.$$

As ω tends to 0, ω/k tends either to the value u_1 or to u_2 (equations (8) and (9)). In the latter case, γ' and μ' tend to γ and μ, with their values as in the present section. Thus, at the limit

$$h_1' = \begin{pmatrix} \mathrm{i}\gamma\mu m_t \\ -\mathrm{i}\mu m_x \\ 0 \end{pmatrix} \tag{61}$$

$$m_1' = \begin{pmatrix} -\mathrm{i}\gamma\mu m_t \\ \mathrm{i}\gamma\mu m_x \\ 0 \end{pmatrix} \tag{62}$$

h_1' is proportional to h_1 if the determinant

$$\begin{vmatrix} \mathrm{i}\gamma\mu m_t & \mu m_x \\ -\mathrm{i}\mu m_x & (1+\alpha) m_t \end{vmatrix} = \mathrm{i}\mu[\mu m_x^2 + \gamma(1+\alpha)m_t^2] \tag{63}$$

is zero. This is equation (39). In the same way, m_1' is proportional to m_1. Thus this propagation mode can truly be considerated as a limiting case of the small-amplitude oscillating wave, as both wavenumber and frequency tend to zero.

It is not possible to describe Nakata's mode in the same way by an analogous limit. Let ω and k tend to zero, keeping ω/k close to $u_1 = \sqrt{\alpha/1+\alpha}$. Then μ' tends to zero and, thus, m_1' and h_1' vanish. Therefore, we choose other eigenvectors, say $m_1'' = m_1'/\mu$, $h_1'' = h_1'/\mu'$. The asymptotic value for μ' can easily be found from the dispersion relation (7) to be

$$\mu' \sim \frac{\gamma}{(1+\alpha)m_t^2}\omega^2. \tag{64}$$

Thus,

$$h_{1,z}'' = \frac{\gamma'\omega}{\mu'} \sim \frac{(1+\alpha)m_t^2}{\omega} \tag{65}$$

and, thus, $h_{1,z}''$ tends to infinity, while $h_{1,x}''$ and $h_{1,y}''$ have finite limits. Hence, Nakata's solitonic propagation mode cannot be considered as a limiting case of a small-amplitude oscillating wave, in contrast to the KdV propagation mode.

4. Equations that describe the interaction of two solitary waves

The two propagation modes described respectively in sections 2 and 3 may interact. In this section, we derive the equations that govern this interaction by using a multiscale expansion. Naturally, the time, space and amplitude scales at which the interaction occurs are not the same as previously: self-interaction of one wave occurs at one scale, self-interaction of the other wave at a second scale, and interaction of both waves at a third scale.

In order to study an interaction between two waves with different group velocities, we must consider time and space scales of the same order. We define

$$\begin{cases} \xi = \varepsilon^p x \\ \tau = \varepsilon^p t \end{cases} \tag{66}$$

where ε is a small parameter and p a positive integer. We expand $\boldsymbol{H}$ and $\boldsymbol{M}$ as in (11) and carry it over to the system (3) and (4). After some trials, we see that the correct choice for p is $p = 2$. Let us put $\varepsilon^2 = \varepsilon'$, and identify ε' (rather than ε) to the small parameter ε used in sections 2 and 3. This way the space scales are the same in all cases, and we can forget that, in sections 2 and 3, we considered a second time scale of higher order (beside the main time scale, of same order as the space scale, involved by the Galilean transform in (10)).

The KdV equation for the propagation mode with speed $\sqrt{\alpha + \sin^2\varphi/(1+\alpha)}$ appears with a very low amplitude scale for the waves: it occurs for the term of order ε'^2 in the expansion of $\boldsymbol{H}$ and $\boldsymbol{M}$. The mKdV equation was obtained by Nakata for higher-intensity waves, i.e. for the mode with speed $\sqrt{\alpha/1+\alpha}$. It concerns the term of order ε'^1. We have noticed that the term of order ε'^0 varies also, but, as written above, this does not characterize the amplitude of the wave. The interaction equations will be obtained for higher intensities: the terms of order $\varepsilon = \varepsilon'^{1/2}$. Thus, the two modes should, separately, have a highly nonlinear behaviour at this scale.

Let us solve system (3) and (4) through the expansion (11) and (66), using the boundary conditions (12) and (13), order by order. At order ε^0, we find

$$\begin{cases} \boldsymbol{H}_0 = \lambda(\xi,\tau)\boldsymbol{M}_0 \\ H_0^x + M_0^x = (1+\alpha)m_x \\ \partial_\tau^2(H_0^s + M_0^s) = \partial_\xi^2 H_0^s \qquad s = y, z. \end{cases} \tag{67}$$

System (67) does not exclude *a priori* a solution with a varying M_0^x. However, in the previous sections M_0^x was constant, therefore, and for the sake of simplicity, we will assume that this is also the case here. Thus λ is also constant: $\lambda \equiv \alpha$, and the third equation in (67) reduces to

$$(\partial_\tau^2 - V_0^2\partial_\xi^2)M_0^s = 0 \qquad s = y, z \tag{68}$$

with

$$V_0 = \sqrt{\frac{\alpha}{1+\alpha}}. \tag{69}$$

M_0^y and M_0^z may propagate with the speed V_0, corresponding to a wave of Nakata's mode. Because this wave appears at order ε^0 and we use a perturbative calculus, it is not affected by the following terms. However, the reaction of the second wave on this wave will affect the term of order ε of the same wave although it cannot appear on this first term.

At order ε^1, we find that there exists a function $f = f(\xi,\tau)$ such that

$$H_1^x = -M_1^x = \frac{m_x}{1+\alpha}f \tag{70}$$

$$M_1^s = \frac{1}{\alpha}(H_1^s - fM_0^s) \qquad s = y, z. \tag{71}$$

Furthermore, we obtain the evolution equations for H_1^y, H_1^z:

$$(\partial_\tau^2 - V_0^2\partial_\xi^2)H_1^s = \frac{1}{1+\alpha}\partial_\tau^2 M_0^s \qquad s = y, z. \tag{72}$$

At order ε^2, we first find the condition

$$\partial_\tau \boldsymbol{M}_0 = -\boldsymbol{M}_0 \wedge [\boldsymbol{H}_2 - \alpha\boldsymbol{M}_2] - \boldsymbol{M}_1 \wedge \boldsymbol{H}_1. \tag{73}$$

Using (70) and (71) and taking the dot product of (73) with $\boldsymbol{M}_0$, we find that $\|\boldsymbol{M}_0\|$ is a constant, as has already been proved in section 3. The solution to equation (68) is, for $s = y, z$,

$$M_0^s = m_0^{s,+}(\xi - V_0\tau) + m_0^{s,-}(\xi + V_0\tau) \tag{74}$$

where $m_0^{y,+}$, $m_0^{y,-}$, $m_0^{z,+}$, $m_0^{z,-}$, are four arbitrary functions. Introducing this solution into the condition $(M_0^y)^2 + (M_0^z)^2 = m_t^2$, we see that either $m_0^{y,+} \equiv m_0^{z,+} \equiv 0$, or $m_0^{y,-} \equiv m_0^{z,-} \equiv 0$. Without loss of generality, we choose this latter case. Thus, we define a function θ such that

$$\begin{cases} M_0^y = m_t \cos\theta \\ M_0^z = m_t \sin\theta \\ \theta = \theta(\xi - V_0\tau). \end{cases} \tag{75}$$

Putting

$$\begin{aligned} M_j^\perp &= M_j^y + \mathrm{i} M_j^z \\ H_j^\perp &= H_j^y + \mathrm{i} H_j^z \end{aligned} \tag{76}$$

for each j, we see that, using (75), expressions (70), (71) and (72) can be written as

$$M_1^\perp = \frac{1}{\alpha} H_1^\perp - \frac{m_t}{\alpha} f \mathrm{e}^{\mathrm{i}\theta} \tag{77}$$

$$(\partial_\tau^2 - V_0^2 \partial_\xi^2) H_1^\perp = \frac{m_t}{1+\alpha} \partial_\tau^2 (f \mathrm{e}^{\mathrm{i}\theta}). \tag{78}$$

Equation (73) can be written as

$$\boldsymbol{M}_0 \wedge \left[-\partial_\tau \theta \boldsymbol{e}_x + [\boldsymbol{H}_2 - \alpha \boldsymbol{M}_2] - \frac{f}{\alpha} \boldsymbol{H}_1 \right] = 0. \tag{79}$$

Thus, there exists a function $h = h(\xi, \tau)$ such that

$$-\partial_\tau \theta \boldsymbol{e}_x + [\boldsymbol{H}_2 - \alpha \boldsymbol{M}_2] - \frac{f}{\alpha} \boldsymbol{H}_1 = h \boldsymbol{M}_0. \tag{80}$$

We have

$$H_2^x = -M_2^x = \frac{1}{1+\alpha} \partial_\tau \theta + \frac{m_x}{\alpha(1+\alpha)^2} f^2 + \frac{m_x}{1+\alpha} h \tag{81}$$

and

$$(\partial_\tau^2 - V_0^2 \partial_\xi^2) H_2^\perp = \partial_\tau^2 \left(\frac{1}{\alpha(1+\alpha)} f H_1^\perp + \frac{m_t}{1+\alpha} h \mathrm{e}^{\mathrm{i}\theta} \right). \tag{82}$$

At order ε^3, we have the equation

$$\partial_\tau \boldsymbol{M}_1 = -\boldsymbol{M}_0 \wedge [\boldsymbol{H}_3 - \alpha \boldsymbol{M}_3] - [\boldsymbol{M}_1 \wedge \boldsymbol{H}_2 + \boldsymbol{M}_2 \wedge \boldsymbol{H}_1]. \tag{83}$$

Taking the dot product of equation (83) with $\boldsymbol{M}_0$, we obtain, after some computation,

$$\begin{aligned} &\frac{-(\alpha m_x^2 + (1+\alpha) m_t^2)}{\alpha(1+\alpha)} (\partial_\tau f) + \frac{m_t}{\alpha} (\cos\theta (\partial_\tau H_1^y) + \sin\theta (\partial_\tau H_1^z)) \\ &\qquad = \frac{m_t}{\alpha} (\partial_\tau \theta)(\sin\theta H_1^y - \cos\theta H_1^z). \end{aligned} \tag{84}$$

Elimination of f between equations (84) and (78) leads to the system we seek. We define the two quantities

$$\begin{cases} \Psi = \sin\theta H_1^z + \cos\theta H_1^y \\ \Phi = \cos\theta H_1^z - \sin\theta H_1^y. \end{cases} \tag{85}$$

That is,

$$\Xi = \Psi + \mathrm{i}\Phi = H_1^\perp \mathrm{e}^{-\mathrm{i}\theta} \tag{86}$$

Ψ is a transversal component of the field $\boldsymbol{H}_1$, parallel to $\binom{M_0^y}{M_0^z}$, the component of the field $\boldsymbol{M}_0$ in the plane perpendicular to the propagation direction. Φ is the component of $\boldsymbol{H}_1$ perpendicular to both the propagation direction and the field $\boldsymbol{M}_0$. Equation (84) can be integrated once to yield

$$f = \frac{m_t}{m^2 V_1^2}\Psi \tag{87}$$

where $m^2 = m_x^2 + m_t^2$, and

$$V_1 = \sqrt{\frac{\alpha + \sin^2\varphi}{1+\alpha}} \tag{88}$$

where V_1 is the velocity of the wave of the KdV mode.

Using (87), and definitions (85) and (86), equation (78) can be written as

$$(\partial_\tau^2 - V_0^2\partial_\xi^2)\Xi e^{i\theta} = \left(1 - \frac{V_0^2}{V_1^2}\right)\partial_\tau^2\Psi e^{i\theta}. \tag{89}$$

Equation (89) can be reduced to the system of two real equations given as

$$\left[\partial_\xi^2 - \frac{1}{V_1^2}\partial_\tau^2 - \left(1 - \frac{V_0^2}{V_1^2}\right)\theta'^2\right]\Psi = 2\theta'\left(\partial_\xi + \frac{1}{V_0}\partial_\tau\right)\Phi \tag{90}$$

$$\left[\partial_\xi^2 - \frac{1}{V_0^2}\partial_\tau^2\right]\Phi = -\left[2\theta'\left(\partial_\xi + \frac{V_0}{V_1^2}\partial_\tau\right) + \left(1 - \frac{V_0^2}{V_1^2}\right)\theta''\right]\Psi. \tag{91}$$

Note that $V_1 > V_0 > 0$ for $\varphi \neq 0, \pi$, and thus

$$0 < \frac{V_0^2}{V_1^2} < 1. \tag{92}$$

System (90) and (91) is a linear hyperbolic system with varying coefficients depending on the function $\theta = \theta(\xi - V_0\tau)$.

Φ propagates at velocity V_0, thus belongs to Nakata's mode. Ψ, propagating with velocity V_1, represents the KdV mode. This last feature can be easily verified by setting θ constant ($\theta = 0$) in system (90) and (91). Then

$$\left[\partial_\xi^2 - \frac{1}{V_1^2}\partial_\tau^2\right]\Psi = 0 \tag{93}$$

$$\left[\partial_\xi^2 - \frac{1}{V_0^2}\partial_\tau^2\right]\Phi = 0. \tag{94}$$

We choose the following particular solution to system (93) and (94):

$$\begin{cases} \Psi = \Psi(\xi - V_1\tau) \\ \Phi = 0. \end{cases} \tag{95}$$

Then, using (87), (85), and (70) and (71), we find

$$\boldsymbol{H}_1 = \begin{pmatrix} \dfrac{m_x m_t}{(1+\alpha)m^2V_1^2} \\ 1 \\ 0 \end{pmatrix}\Psi \tag{96}$$

using the relation

$$\mu = \frac{\sin^2\varphi}{V_1^2} \tag{97}$$

and putting

$$g = \frac{1}{(1+\alpha)m_t}\Psi. \tag{98}$$

(96) is identical to equation (37) which gives the same term, in the frame where we derived the KdV equation for the function g. Thus Ψ represents the propagation mode studied in section 3.

Let us now show that Φ can be considered as a correction term to θ. We look for solutions to system (90) and (91) for which $\Psi \equiv 0$. We must have

$$\theta' \left(\partial_\xi + \frac{1}{V_0}\partial_\tau\right)\Phi = 0 \tag{99}$$

and

$$\left[\partial\xi^2 - \frac{1}{V_0^2}\partial_\tau^2\right]\Phi = 0. \tag{100}$$

We obtain a solution with

$$\Phi = \Phi(\xi - V_0\tau). \tag{101}$$

$H_1^x = 0$ because $\Psi = 0$, and

$$\begin{cases} H_1^y = -\Phi\sin\theta \\ H_1^z = \Phi\cos\theta. \end{cases} \tag{102}$$

Let $H^\perp = H^y + \mathrm{i}H^z$,

$$H^\perp = H_0^\perp + \varepsilon H_1^\perp + \mathrm{O}(\varepsilon^2). \tag{103}$$

Using (75) and (102),

$$H^\perp = m_t \mathrm{e}^{\mathrm{i}\theta}\left(1 + \mathrm{i}\varepsilon\frac{\Phi}{m_t} + \mathrm{O}(\varepsilon^2)\right). \tag{104}$$

We can write this as

$$H^\perp = m_t \exp\left[\mathrm{i}\left(\theta + \varepsilon\frac{\Phi}{m_t} + \mathrm{O}(\varepsilon^2)\right)\right]. \tag{105}$$

Thus $\varepsilon\phi/m_t$ is the correction to order ε to the wave θ.

System (90) and (91) describes the interaction between the two waves Ψ and $\theta + \varepsilon\Phi/m_t$. This interaction is by itself nonlinear, but the use of the multiscale expansion forced us to treat θ as given data, and to drop correction terms of order higher than $\varepsilon\Phi/m_t$, and therefore the obtained system is linearized. The dependence on the wave θ is, however, nonlinear.

5. A resolution of the interaction system

5.1. Introduction

Defining $\lambda = V_0^2/V_1^2$ and $T = V_0\tau$, system (90) and (91) can be written:

$$(\partial\xi^2 - \lambda\partial_T^2)\Psi = f \tag{106}$$

$$(\partial\xi^2 - \partial_T^2)\Phi = g \tag{107}$$

$$f = (1-\lambda)\theta'^2\Psi + 2\theta'(\partial_\xi + \partial_T)\Phi \tag{108}$$

$$g = -[2\theta'(\partial_\xi + \lambda\partial_T) + (1-\lambda)\theta'']\Psi \tag{109}$$

$\lambda \in]0,1[$ is a constant, and $\theta = \theta(\xi - T)$ a given function. We intend to solve this system for the following initial conditions: we assume that, at $T = 0$, $\Psi(\xi, 0)$, $\partial_T \Psi(\xi, 0)$, $\Phi(\xi, 0)$, $\partial_T \Phi(\xi, 0)$ are given.

First we put it into an integral form. Under the general physical assumption that, at $T = 0$, the correction term Φ is zero, and θ and Ψ are separated, Ψ is defined by equation (123) with the operator $\mathcal{L}$ given by (129) and Φ by equation (115), with the operator $\mathcal{L}_1$ given by (116). Then, we compute explicit solutions in the particular case where the initial conditions are the following: θ is an even continuous function that approaches the distribution $2\pi\delta(\xi - T)$, and, before the interaction, $\Psi = g(\xi - vT)$, where g is a continuous localized function. We still assume that, at this time, θ and Ψ are separated, thus $g(0) = 0$, and Φ is zero. Computation of Ψ leads to expression (155). A transmitted and a reflected wave arise, and transmission and reflexion coefficients are computed. Finally, the correction term Φ to θ is computed; it is given by formulae (177), (178) and (181). This term is interpreted as the appearance of a reflected wave, and a phase shift.

5.2. *Integral form of the interaction equations*

We put

$$\lambda = \frac{1}{v^2} \qquad v = \frac{V_1}{V_0} > 1. \tag{110}$$

Using the change of variables

$$\begin{cases} X = \xi - vT \\ Y = \xi + vT \end{cases} \tag{111}$$

and acting as if f were a known function, we can write equation (106) in the integral form:

$$\Psi(\xi, T) = -\frac{v}{2} \int_0^T \mathrm{d}T' \int_{\xi - v(T-T')}^{\xi + v(T-T')} \mathrm{d}\xi' \, f(\xi', T') + \frac{1}{2v} \int_{\xi - vT}^{\xi + vT} \mathrm{d}\xi' \, \partial_T \Psi(\xi', 0)$$
$$+ \tfrac{1}{2}[\Psi(\xi - vT, 0) + \Psi(\xi + vT, 0)]. \tag{112}$$

In an analogous way, equation (107) can be written as

$$\Phi(\xi, T) = -\tfrac{1}{2} \int_0^T \mathrm{d}T' \int_{\xi - (T-T')}^{\xi + (T-T')} \mathrm{d}\xi' \, g(\xi', T') + \tfrac{1}{2} \int_{\xi - T}^{\xi + T} \mathrm{d}\xi' \, \partial_T \Phi(\xi', 0)$$
$$+ \tfrac{1}{2}[\Phi(\xi - T, 0) + \Phi(\xi + T, 0)]. \tag{113}$$

Because Φ is physically a correction to θ, we can assume, without loss of generality from the physical point of view, that

$$\Phi(\xi, 0) = 0 \qquad \text{and} \qquad \partial_T \Phi(\xi, 0) = 0 \qquad \text{for all } \xi. \tag{114}$$

Thus Φ can be expressed as a function of Ψ:

$$\Phi = \mathcal{L}_1 \Psi \tag{115}$$

with

$$\mathcal{L}_1 u(\xi, T) = \tfrac{1}{2} \int_0^T \mathrm{d}T' \int_{\xi - T + T'}^{\xi + T - T'} \mathrm{d}\xi' \, [2\theta'(\partial_\xi + \lambda \partial_T) + (1 - \lambda)\theta''] u(\xi', T') \tag{116}$$

for any function u. Let us define

$$\Psi_0(\xi, T) = \frac{1}{2v} \int_{\xi - vT}^{\xi + vT} \mathrm{d}\xi' \, \partial_T \Psi(\xi', 0) + \tfrac{1}{2}[\Psi(\xi - vT, 0) + \Psi(\xi + vT, 0)] \tag{117}$$

Ψ_0 is the solution of the equation

$$(\partial_\xi^2 - \lambda\partial_T^2)\Psi_0 = 0 \tag{118}$$

with the initial conditions

$$\Psi_0(\xi,0) = \Psi(\xi,0) \qquad \text{and} \qquad \partial_T\Psi(\xi,0) = \partial_T\Psi(\xi,0) \qquad \text{for all } \xi. \tag{119}$$

We also define $\mathcal{L}_2$ and $\mathcal{L}_3$ such that

$$\mathcal{L}_2 u(\xi,T) = -\frac{v}{2}\int_0^T \mathrm{d}T' \int_{\xi-v(T-T')}^{\xi+v(T-T')} \mathrm{d}\xi'\,(1-\lambda)\theta'^2 u(\xi',T') \tag{120}$$

$$\mathcal{L}_3 u(\xi,T) = -\frac{v}{2}\int_0^T \mathrm{d}T' \int_{\xi-v(T-T')}^{\xi+v(T-T')} \mathrm{d}\xi'\,2\theta'(\partial_\xi+\partial_T)u(\xi',T') \tag{121}$$

for any function u. Using this notation, equation (112), with f given by (108), reads as

$$\Psi = \mathcal{L}_2\Psi + \mathcal{L}_3\Phi + \Psi_0. \tag{122}$$

Using (115), we get

$$\Psi = \mathcal{L}\Psi + \Psi_0 \tag{123}$$

with

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3\mathcal{L}_1. \tag{124}$$

Then the resolution of equation (123) is theoretically easy: we construct a recurrent sequence $\Psi_0, \Psi_1, \Psi_2, \ldots$, such that Ψ_0 is the given function and, for each n, $\Psi_{n+1} = \mathcal{L}\Psi_n$. Then (if we assume that the series converges):

$$\Psi = \sum_{n=0}^{\infty}\Psi_n \tag{125}$$

is the solution to equation (123) [18].

To obtain a suitable expression for $\mathcal{L}$, we first derivate $\Phi = \mathcal{L}_1\Psi$ to obtain

$$(\partial_\xi+\partial_T)\Phi(\xi,T) = -\int_0^T \mathrm{d}T'\,g(\xi+T-T',T'). \tag{126}$$

Then an integration by parts removes the term proportional to θ'' in the integrand of (126). We have

$$\begin{aligned}\theta''(\xi+T-2T')\Psi(\xi+T-T',T') &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}T'}(\theta'(\xi+T-2T')\Psi(\xi+T-T',T'))\\ &\quad+\tfrac{1}{2}\theta'(\xi+T-2T')(-\partial_\xi+\partial_T)\Psi(\xi+T-T',T').\end{aligned} \tag{127}$$

Thus

$$\begin{aligned}(\partial_\xi+\partial_T)\Phi(\xi,T) &= \frac{-(1-\lambda)}{2}[\theta'(\xi-T)\Psi(\xi,T) - \theta'(\xi+T)\Psi(\xi+T,0)]\\ &\quad+\tfrac{1}{2}\int_0^T \mathrm{d}T'\,\theta'(\xi+T-2T')[(3+\lambda)\partial_\xi + (3\lambda+1)\partial_T]\Psi(\xi+T-T',T').\end{aligned} \tag{128}$$

In the expression for f, the term proportional to θ'^2 cancels the term proportional to $\theta'(\xi-T)$ of formula (128). If we assume that the two waves θ and Ψ have separated supports at $T=0$, we have $\theta'(\xi+T)\Psi(\xi+T,0)\equiv 0$. Thus we obtain

$$\begin{aligned}\mathcal{L}\Psi(\xi,T) &= -\frac{v}{2}\int_0^T \mathrm{d}T' \int_{\xi-v(T-T')}^{\xi+v(T-T')} \mathrm{d}\xi'\,\theta'(\xi'-T')\int_0^{T'} \mathrm{d}T''\,\theta'(\xi'+T'-2T'')\\ &\quad\times[(3+\lambda)\partial_\xi + (3\lambda+1)\partial_T]\Psi(\xi'+T'-T'',T'').\end{aligned} \tag{129}$$

5.3. Explicit resolution for particular initial data

The only hypotheses that we have made up to this point are: Φ is zero at $T = 0$ and θ and Ψ have disjoint supports at $T = 0$. To go further in the computation, we need additional hypotheses on the shape of the input waves Ψ_0 and θ. θ describes a Nakata-mode wave. Therefore the following choice seems particularly interesting: θ' should look like a soliton of mKdV, which is given by equation (28). It has a localized bell shape, and vanishes quickly at both $+\infty$ and $-\infty$ in time and space. In the same way, Ψ_0 should look like a soliton of the KdV, which is given by equation (57), and has analogous characteristics. In order to perform an explicit calculus, it seems useful to model the bell shape of these waves by simple initial data, such as delta functions, for θ' and Ψ_0. This would describe an asymptotic case, as the typical length of the solitons is very small in regard to the space scale at which the interaction occurs.

Let us show that this requirement is consistent with the scalings. Consider two given *input solitary waves θ' and Ψ_0. A scale parameter ε_0 is defined by their amplitude. The* mKdV equation (27) concerns a first-order quantity, thus for the mKdV mode, the parameter ε in (10) is equal to ε_0, and the typical length of a soliton of this mode is

$$L_{\mathrm{mKdV}} = \frac{L}{\varepsilon_0} \tag{130}$$

(L being a length of order unity). The KdV equation (42) concerns a second-order quantity, thus in this case $\varepsilon = \sqrt{\varepsilon_0}$ and the typical length of the KdV soliton will be

$$L_{\mathrm{KdV}} = \frac{L}{\sqrt{\varepsilon_0}}. \tag{131}$$

The interaction system (90) and (91) concerns first-order quantities, but the scaling is different (66), thus $\varepsilon = \varepsilon_0$, but the typical length of the interaction process is

$$L_{int} = \frac{L}{\varepsilon_0^2}. \tag{132}$$

Therefore, while $\varepsilon_0 \ll 1$,

$$L_{\mathrm{int}} \gg L_{\mathrm{mKdV}} \gg L_{\mathrm{KdV}}. \tag{133}$$

The choice of delta functions for the initial data is therefore physically consistent. However, for mathematical reasons, it is not possible to make this choice directly, and we must reason on continuous functions. Furthermore, we want to make the minimal assumptions necessary to perform an explicit calculus. In a first stage, we only assume that θ' is a regular function localized in the interval $(\xi - T) \in [-b, b]$, with $b > 0$.

In a second stage, we will let b tend to zero. Furthermore, we will assume that θ' is even, and that $\int_{\mathbb{R}} \theta' = 2\pi$, which is true for a soliton of the mKdV equation (27). We choose for Ψ_0 an arbitrary regular and localized function $g(\xi - vT)$, with $g(0) = 0$ (no further assumption is needed). For mathematical reasons, it is very difficult to make another choice. Thus, physically, we assume that the spatial extension of the Nakata-mode wave θ' is small with respect to the space scale of the interaction, but also to the variation scale of the second wave Ψ. According to the inequalities (133), the first assumption is consistent with the interpretation of the incident wave θ' as a soliton of Nakata's mode. The second assumption implies that we observe the propagation mode that can support KdV solitons, at a space scale which is very large with respect to the size of these solitons. The solitary wave Ψ under consideration will therefore be a long pulse, containing a very large number of solitons, and the present treatment of the interaction system will disregard the solitonic structure of this pulse, and act only on an averaged amplitude of the wave.

Let us first evaluate $\mathcal{L}u$ for an arbitrary function u. $\mathcal{L}u$ is an integral over a domain $\mathcal{D}$ in the (ξ', T', T'') space, that is a tetrahedron of vertices $(\xi, T, 0)$, (ξ, T, T), $(\xi + vT, 0, 0)$, $(\xi - vT, 0, 0)$. We use the change of variables

$$\begin{cases} x = \xi' - T' \\ y = \xi' + T' - 2T'' \\ z = T''. \end{cases} \tag{134}$$

Then

$$\mathcal{L}u = -\frac{v}{4} \iiint_{\mathcal{D}'} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \theta'(x)\theta'(y)h \tag{135}$$

where

$$h = [(3+\lambda)\partial_\xi + (3\lambda + 1)\partial_T]u(\xi' + T' - T'', T'') \tag{136}$$

and $\mathcal{D}'$ is $\mathcal{D}$ transformed by (134). Using the fact that θ' is localized, we restrict the integration domain to $\mathcal{D}_1 = \mathcal{D}' \cap \mathcal{D}''$, where $\mathcal{D}''$ is the set of the points (x, y, z) for which $-b \leqslant x \leqslant b$ and $-b \leqslant y \leqslant b$. The transform of $\mathcal{D}_1$ into the (ξ', T', T'') space is a tube about a straight line D directed by vector $(1, 1, 1)$. $\mathrm{D} \cap \mathcal{D}$ is a segment $[0, \mathrm{A}]$ where $\mathrm{A} = A(\xi, T)\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)$, with

$$A(\xi, T) = \begin{cases} 0 & \text{if } |\xi| \geqslant vT \\ \dfrac{vT - \xi}{v - 1} & \text{if } T \leqslant \xi \leqslant vT \\ \dfrac{vT + \xi}{v + 1} & \text{if } -vT \leqslant \xi \leqslant T. \end{cases} \tag{137}$$

We assume that we observe the phenomenon at a space and time scale which is large with respect to the duration of the pulse θ', thus b takes very small values. After some geometrical work, we see that, neglecting the terms of higher order in b,

$$\mathcal{L}u \simeq -\frac{v}{4} \int_{-b}^{b} \mathrm{d}x \int_{x}^{b} \mathrm{d}y \int_{0}^{A(\xi,T)} \mathrm{d}z\, \theta'(x)\theta'(y)h. \tag{138}$$

Now we assume that Ψ_0 has the form

$$\Psi_0(\xi, T) = g(\xi - vT) \tag{139}$$

where g is a regular and localized function. Recall that at $t = 0$, the supports of the two waves are assumed to be disjoint. Thus we have

$$g(0) = 0. \tag{140}$$

We intend to compute $\Psi_1 = \mathcal{L}\Psi_0$. Using (136) and (134), we compute h for $u = \Psi_0$:

$$h = -\frac{(v-1)^3}{v^2} g' \left[\frac{v+1}{2} y - \frac{(v-1)}{2}(2z + x) \right]. \tag{141}$$

As b becomes very small, only values of x and y very close to zero are to be taken into account in (138). Thus, h being continuous, we can replace $h(x, y, z)$ by $h(x = 0, y = 0, z)$, and equation (138) becomes

$$\mathcal{L}u \simeq -\frac{v}{4} \iint_{\substack{-b \leqslant x \leqslant b \\ -b \leqslant y \leqslant b \\ x \leqslant y}} \mathrm{d}x\, \mathrm{d}y\, \theta'(x)\theta'(y) \int_0^{A(\xi,T)} h(x = 0, y = 0, z)\, \mathrm{d}z. \tag{142}$$

Let us call I the first integral. We note that a permutation of x and y does not change its value, thus

$$2I = \iint_{\substack{-b\leqslant x\leqslant b\\ -b\leqslant y\leqslant b}} \mathrm{d}x\,\mathrm{d}y\,\theta'(x)\theta'(y) = \left(\int_{-b}^{b} \theta'(x)\,\mathrm{d}x\right)^2 \tag{143}$$

and

$$\int_{-b}^{b} \theta'(x)\,\mathrm{d}x = \int_{\mathbb{R}} \theta'(x)\,\mathrm{d}x = \theta(+\infty) - \theta(-\infty). \tag{144}$$

At infinity, there are no waves thus the direction of M_0 is defined by the exterior field, thus θ is zero modulo 2π at $\pm\infty$. For the sake of simplicity, we choose $\int_{\mathbb{R}} \theta'(x)\,\mathrm{d}x = 2\pi$, which corresponds to one soliton of Nakata's type. Thus $I = 2\pi^2$. The second integral in (142) is immediately computed using (141) and (140). Finally

$$\mathcal{L}\Psi_0 \simeq \frac{-\pi^2(v-1)^2}{2v} g(-(v-1)A(\xi, T)). \tag{145}$$

Thus we have

$$\Psi_1 \propto g(-(v-1)A(\xi, T)). \tag{146}$$

Let us therefore compute $\mathcal{L}u$ for a more general function $u(\xi, T) = G(A(\xi, T))$, G being a regular localized function with $G(0) = 0$. From (136) we get

$$h = [(3+\lambda)(\partial_\xi A) + (3\lambda + 1)(\partial_T A)]G'(A(\xi_0, T_0)) \tag{147}$$

with

$$(\xi_0, \tau_0) = (\tfrac{1}{2}(x + 2z + y), \tfrac{1}{2}(x + 2z - y)). \tag{148}$$

This leads to some difficulty because as $x = y = 0$, h does not exist. A is not differentiable on the line $\xi = T$. The derivatives of $A(\xi, T)$ have different expressions for $\xi_0 > T_0$ $(y > 0)$ and for $\xi_0 < T_0$ $(y < 0)$. Thus

$$h = \begin{cases} \left(\dfrac{v-1}{v}\right)^2 G'(A(\xi_0, T_0)) & \text{for } y > 0 \\[2ex] \left(\dfrac{v+1}{v}\right)^2 G'(A(\xi_0, T_0)) & \text{for } y < 0. \end{cases} \tag{149}$$

We divide the integration in formula (138) into two parts corresponding to $y > 0$ and $y < 0$. We assume that θ' is an even function. This is true if θ' has the shape of a mKdV soliton. Thus

$$\iint_{\substack{x\leqslant y\\ y>0}} \mathrm{d}x\,\mathrm{d}y\,\theta'(x)\theta'(y) = \tfrac{3}{2}\pi^2 \tag{150}$$

and

$$\iint_{\substack{x\leqslant y\\ y<0}} \mathrm{d}x\,\mathrm{d}y\,\theta'(x)\theta'(y) = \tfrac{1}{2}\pi^2. \tag{151}$$

We thus find that

$$\mathcal{L}u \simeq -\frac{\pi^2}{2v}(v^2 - v + 1)G(A(\xi, T)). \tag{152}$$

We can now evaluate the solution Ψ given by (125):

$$\Psi = \Psi_0 + \sum_{n=0}^{\infty} \left[-\frac{\pi^2}{2v}(v^2 - v + 1)\right]^n \Psi_1. \tag{153}$$

The series does not converge. However, if the assumption $\int_{\mathbb{R}} \theta' = 2\pi$ is replaced by a less physical one $\int_{\mathbb{R}} \theta' = 2\rho$, ρ arbitrary, it does converge if

$$|\rho| < \sqrt{\frac{2v}{v^2 - v + 1}}. \tag{154}$$

We have only to verify that the solution obtained can be continued in a solution to equation (123) even as $\rho = \pi$. This is easily done by direct computation using (145) and (152).

Then

$$\Psi(\xi, T) = g(\xi - vT) - \frac{\pi^2(v-1)^2}{2v} \frac{1}{1 + (\pi^2/2v)(v^2 - v + 1)} g(-(v-1)A(\xi, T)) \tag{155}$$

$A(\xi, T)$ is given by (137).

For $T \leqslant \xi \leqslant vT$, $-(v-1)A(\xi, T) = \xi - vT$, and we can write

$$\Psi(\xi, T) = \mathcal{T} g(\xi - vT). \tag{156}$$

$\mathcal{T}$ plays the role of a transmission coefficient and has the expression

$$\mathcal{T} = 1 - \frac{\pi^2(v-1)^2}{2v} \frac{1}{1 + (\pi^2/2v)(v^2 - v + 1)} \tag{157}$$

or

$$\mathcal{T} = \frac{v(2 + \pi^2)}{2v + \pi^2(v^2 - v + 1)}. \tag{158}$$

As $-vT \leqslant \xi < T$, $-(v-1)A(\xi, T) = -\frac{v-1}{v+1}(\xi + vT)$, thus we can write

$$\Psi = g(\xi - vT) - \mathcal{R} g\left(-\frac{v-1}{v+1}(\xi + vT)\right). \tag{159}$$

The term $-\mathcal{R} g(-\frac{v-1}{v+1}(\xi + vT))$ represents a reflected wave, and $\mathcal{R}$ is a reflection coefficient. We have

$$\mathcal{R} = \frac{\pi^2(v-1)^2}{2v + \pi^2(v^2 - v + 1)} \tag{160}$$

and

$$\mathcal{R} + \mathcal{T} = 1. \tag{161}$$

5.4. *Computation of the correction term Φ to the wave θ*

Now we have to compute the second wave Φ. Φ can be expressed as a function of Ψ through equations (115) and (116):

$$\Phi(\xi, T) = \tfrac{1}{2} \int_0^T \mathrm{d}T' \int_{\xi - T + T'}^{\xi + T - T'} \mathrm{d}\xi' \, (2\theta'(\partial_\xi + \lambda \partial_T) + (1 - \lambda)\theta'')\Psi(\xi', T'). \tag{162}$$

An integration by parts removes the term in θ'', and Φ can be divided into three terms:

$$\Phi(\xi, T) = \frac{1}{2}\Phi_1(\xi, T) + \frac{1-\lambda}{2}\Phi_2(\xi, T) - \frac{1-\lambda}{2}\theta'(\xi, T)\Phi_3(\xi, T) \tag{163}$$

with

$$\Phi_1(\xi, T) = \int_0^T \mathrm{d}T' \int_{\xi-T+T'}^{\xi+T-T'} \mathrm{d}\xi'\, \theta'((1+\lambda)\partial_\xi + 2\lambda\partial_T)\Psi(\xi', T') \tag{164}$$

$$\Phi_2(\xi, T) = \int_0^T \mathrm{d}T'\, \theta'(\xi + T - 2T')\Psi(\xi + T - T', T') \tag{165}$$

$$\Phi_3(\xi, T) = \int_0^T \mathrm{d}T'\, \Psi(\xi - T + T', T'). \tag{166}$$

Under the preceding hypothesis, we can replace $\theta'(X)$ in (165) by $2\pi\delta(X)$, thus

$$\Phi_2 \simeq \begin{cases} \pi\Psi\left(\dfrac{\xi+T}{2}, \dfrac{\xi+T}{2}\right) & \text{for } |\xi| < T \\ 0 & \text{otherwise.} \end{cases} \tag{167}$$

To evaluate Φ_1, we first write

$$\Psi(\xi, T) = g(\xi - vT) + g_1(A(\xi, T)) \tag{168}$$

with

$$g_1(u) = -\mathcal{R}g(-(v-1)u). \tag{169}$$

According to (168), the integrand in (164) can be divided into two parts, h_1 and h_2. The first term corresponding to g is evaluated:

$$h_1(\xi', T') = \left(\frac{1-v}{v}\right)^2 g'(\xi' - vT') \tag{170}$$

and replacing $\theta'(X)$ by $2\pi\delta(X)$, we compute the corresponding term $\Phi_{1,1}$ of Φ_1:

$$\Phi_{1,1}(\xi, T) = \begin{cases} -\dfrac{2\pi(v-1)}{v^2} g\left(-(v-1)\dfrac{\xi+T}{2}\right) & \text{if } |\xi| < T \\ 0 & \text{otherwise.} \end{cases} \tag{171}$$

The second term reads:

$$h_2(\xi', T') = \begin{cases} \dfrac{-(v-1)}{v^2} g_1'(A(\xi', T')) & \text{if } T < \xi < vT \\ \dfrac{v+1}{v^2} g_1'(A(\xi', T')) & \text{if } -vT < \xi < T \\ 0 & \text{if } |\xi| > vT. \end{cases} \tag{172}$$

Because $h_2(\xi', T')$ is not continuous on the line $\xi' = T'$, we must use a continuous function θ' and not a delta function. For $|\xi| \leqslant T$, as b tends to zero, the integration domain can be assimilated to the rectangle $[-b, b] \times [0, \frac{\xi+T}{2}]$ in the $(\xi' - T', T')$ plane. Then it is divided into two parts, corresponding to $\xi' > T'$ and to $\xi' < T'$. Each part factorizes, thus we obtain for the corresponding term $\Phi_{1,2}$ of Φ_1:

$$\Phi_{1,2} = \frac{2\pi}{v^2} g_1\left(\frac{\xi+T}{2}\right). \tag{173}$$

Finally,

$$\Phi_1 = \begin{cases} \dfrac{-2\pi}{v^2}(v - 1 + \mathcal{R}) g\left(-\dfrac{(v-1)}{2}(\xi + T)\right) & \text{if } |\xi| \leqslant T \\ 0 & \text{otherwise.} \end{cases} \tag{174}$$

The last term is $\Phi_3(\xi, T)$. Because it appears only with the factor $\theta'(\xi - T)$, which approaches $2\pi\delta(\xi - T)$, we only need to compute $\Phi_3(T, T)$. We have

$$\Phi_3(T, T) = -\frac{1-\mathcal{R}}{v-1}G(-(v-1)T) \tag{175}$$

where $G(T)$ is the function defined by

$$G(T) = \int_0^T g(u)\,\mathrm{d}u. \tag{176}$$

Finally we have

$$\Phi(\xi, T) = -\mathcal{K}\Theta(T - |\xi|)g\left(-(v-1)\frac{\xi+T}{2}\right) + Q(T)\theta'(\xi - T) \tag{177}$$

with

$$\mathcal{K} = \frac{\pi(\pi^2-2)}{2}\frac{(v-1)^2}{v}\frac{1}{2v+\pi^2(v^2-v+1)}. \tag{178}$$

Θ is the Heaviside function and

$$Q(T) = \frac{v+1}{2v^2}TG(-(v-1)T). \tag{179}$$

Note that, for large values of T, $Q(T)$ is a constant. Indeed, if g is localized, we have, for large enough T,

$$G(-(v-1)T) = \int_0^{-\infty} g(u)\,\mathrm{d}u. \tag{180}$$

The part of the support of g belonging to the $T > 0$ does not contribute, because the corresponding wave Ψ is ahead of the wave θ, and is faster, thus they never interact. If we call $\tilde{\Psi} = \int_{-\infty}^0 g$, for T large enough,

$$Q(T) = -\mathcal{Q}\tilde{\Psi} \tag{181}$$

with

$$\mathcal{Q} = \frac{(v+1)}{2v}\frac{2+\pi^2}{2v+\pi^2(v^2-v+1)}. \tag{182}$$

Note that $\mathcal{Q}$ is always positive, and that the sign of $\tilde{\Psi}$ depends on that of g (or Ψ_0), that is the amplitude of the incident wave.

Using (177) and (181), the complete wave of Nakata's mode, $\theta + \varepsilon\Phi/m_t$, can be written:

$$\theta + \varepsilon\frac{\Phi}{m_t} = \theta_\mathrm{r} + \theta_\mathrm{t} \tag{183}$$

with, for T large enough,

$$\theta_\mathrm{t}(\xi, T) = \theta(\xi - T) - \frac{\varepsilon}{m_t}\mathcal{Q}\tilde{\Psi}\theta'(\xi - T) \tag{184}$$

and for $t > |\xi|$

$$\theta_\mathrm{r}(\xi, T) = -\frac{\varepsilon}{m_t}\mathcal{K}g\left(-(v-1)\frac{\xi+T}{2}\right) \tag{185}$$

θ_r is a reflected wave. It has a small amplitude, unless it belongs to the finite-amplitude mode. It has the shape of the incident wave of the KdV mode $\Psi_0(\xi, T) = g(\xi - vT)$.

Neglecting terms of order ε^2, the transmitted wave θ_t can be written:

$$\theta_t \simeq \theta\left(\xi - T - \frac{Q}{m_t}\varepsilon\tilde{\Psi}\right). \tag{186}$$

Thus the wave is delayed for a time $\frac{Q}{m_t}\varepsilon\tilde{\Psi}$, proportional to the integral $\varepsilon\tilde{\Psi}$ of the amplitude of the incident wave of the KdV mode. The delaying time is positive or negative depending on the sign of Ψ. It can be interpreted using equations (96) and (98) and (36) and (37). Let $b_1 = h_1 + m_1$, $B_1 = b_1 g$ is the magnetic induction corresponding to the field $H_1 = h_1 g$ and magnetization density $M_1 = m_1 g$ of equations (36) and (37). It reads:

$$b_1 = \frac{(1+\alpha)m_t}{V^2}((010) \tag{187}$$

b_1^y has the same sign as $B_0^y = H_0^y + M_0^y$. Thus a positive g in (36) and (37), that is a positive Ψ, corresponds to a local increase in the norm of the total induction B, and a negative g—or Ψ—to a propagating 'hole' in this induction. In the former case, the Nakata-mode wave is delayed, while it is brought forward in the latter case.

6. Conclusion

We have first described two solitonic propagation modes of electromagnetic waves in ferromagnets. One of them is governed by the mKdV equation, and was discovered by Nakata. It is a finite-amplitude mode. The second is governed by the mKdV equation and is, in contrast to the former, a small-amplitude mode. Note that the solitonic behaviour does not occur on the same scale for the two modes. At a still larger space scale (assuming the same amplitude scale), the two modes may interact. A partial differential system that describes the interaction has been derived by means of a multiscale expansion.

The small perturbative parameter used in this expansion is related to the amplitude of the wave of the mode that can support KdV solitons. Because the other mode is of finite amplitude, it appears as a zeroth-order term in the perturbative expansion, and therefore cannot be influenced by the KdV wave, which is a term of first order. Thus the reaction on the mKdV wave is described by a third term that corresponds to a small correction to the finite-amplitude wave. Because this wave is split into two terms (an incident wave and the first correction due to the interaction) the interaction system (90) and (91) becomes linear with respect to the two first-order wave components.

Note that we are not dealing here with a soliton interaction in the usual sense, which is the interaction between two solitons with the same propagation mode, described by a two-soliton solution of an integrable equation. Here we study the interaction between two solitary waves which are different in nature. Furthermore, these waves can form solitons only at space scales that differ from the scale on which interaction occurs. Therefore the usual behaviour of a soliton interaction (conservation of the shape and size, and the appearance of a phase shift) is *a priori* not to be expected here.

The interaction system has been solved in the particular case where the duration of the incident pulse of Nakata's mode is very short with respect to the interaction time (or space) scale. This particular case corresponds to a one-solitonic solution of the mKdV equation for the Nakata-mode wave, but disregards the solitonic structure of the wave of the KdV mode, in considering a pulse very long in regard to the size of the solitons. The wave of the KdV mode, which is a small-amplitude mode, is partially reflected on the Nakata-mode wave, which has a finite amplitude. We have computed transmission and reflection coefficients for this process. The other wave is also partially reflected, and is delayed proportionally to the integral of the amplitude of the wave of the KdV mode. This may appear as an analogue of the phase shift in usual solitons interaction, unless the frame is rather different.

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